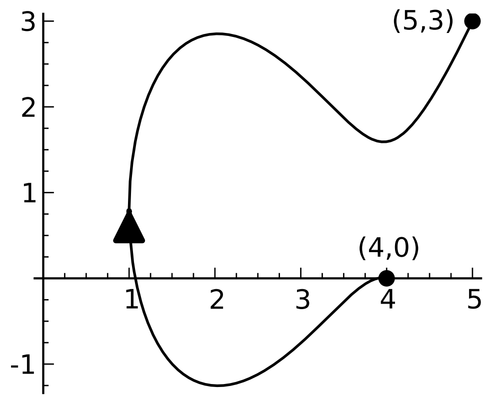
1. (4 points) Let $f = x + xy + y$, and let C be the curve below, with endpoints $(4, 0)$ and $(5, 3)$, and oriented in the clockwise-ish direction.



Determine $\int_C \nabla f \cdot d\mathbf{r}$.

Solution: Using the fundamental theorem of line integrals: $f(5, 3) - f(4, 0) = 19$.

2. (10 points) Let C_1 be the spiral parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 6\pi$. Let C_2 be the line segment from $(1, 0, 6\pi)$ to $(1, 0, 0)$. Let C be C_1 followed by C_2. Determine

$$\int_C \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right\rangle \cdot d\mathbf{r}.$$

Solution: The integral over C_1 is

$$\int_0^{6\pi} \langle -\sin t, \cos t, t \rangle \cdot \langle -\sin t \cos t, 1 \rangle \; dt = \int_0^{6\pi} 1 + t \, dt = 6\pi + 18\pi^2.$$

For C_2 we have $\mathbf{r}(t) = \langle 1, 0, t \rangle$ with t from 6π to 0. So we have

$$\int_{6\pi}^{0} \langle 0, 1, t \rangle \cdot \langle 0, 0, 1 \rangle \, dt = \int_{6\pi}^{0} t \, dt = -18\pi^2.$$

Adding the two, we get 6π for our final answer.

3. (10 points) Let C be the path that goes in a straight line from $(1,0,0)$ to $(0,-1,0)$ to $(0,0,1)$ and back to $(1,0,0)$. Use Stokes' Theorem to set up a double integral that computes
$$\int_C \langle xyz, x+y, x+z\rangle \cdot d\mathbf{r}.$$
Do not evaluate. Your answer should have two variables only and no vectors, looking something like this: $\int_{-}^{-}\int_{-}^{-} \_\_\_\_ \, dx\, dy$.

Solution: If $\mathbf{F}$ is the vector field in the integral, then $\operatorname{curl}\mathbf{F} = \langle 0, xy-1, 1-xz\rangle$. To use Stokes' Theorem, we need to parametrize the triangular surface T, which is part of the plane $x-y+z=1$. We do this with $\mathbf{r}(x,y) = \langle x, y, 1-x+y\rangle$ for (x,y) in the triangle D with vertices $(1,0)$, $(0,-1)$, and $(0,0)$. We calculate $\mathbf{r}_x \times \mathbf{r}_y = \langle 1,-1,1\rangle$. The integral is
$$\begin{aligned}\iint_T \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \langle 0, xy-1, 1-x(1-x+y)\rangle \cdot \langle 1,-1,1\rangle \; dA \\ &= \int_0^1 \int_{x-1}^0 \left(2-2xy+x^2-x\right)\, dy\, dx.\end{aligned}$$
Now we should worry about orientation. Looking down at the triangle from above, we see that the line integral is going clockwise around the triangle, which means we need the normal vector to point down so that it satisfies the right-hand rule. Our normal vector pointed up, which means the correct answer is negative of what we have above.

4. (10 points) Let E be the region above the plane $y+z=-6$, below the plane $x+z=6$, and inside the cylinder $x^2+y^2=9$. Let S be the boundary of E (the sides of the cylinder + the ellipse at the top + the ellipse at the bottom) with the positive (outward) orientation. Calculate
$$\iint_S \langle x^3, z^3, 3y^2z\rangle \cdot d\mathbf{S}.$$

Solution: If $\mathbf{F}$ is the vector field in the integrand, then $\operatorname{div}\mathbf{F} = 3(x^2+y^2)$. By the Divergence Theorem, the integral is equal to
$$\begin{aligned}\iiint_E 3(x^2+y^2)\, dV &= \int_0^{2\pi}\int_0^3\int_{-6-r\sin\theta}^{6-r\cos\theta} 3r^3\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^3 3r^3(12+r\sin\theta-r\cos\theta)\, dr\, d\theta \\ &= \frac{3^5}{4}(24\pi) = (2\pi)(3^6) \text{ or } 1458\pi.\end{aligned}$$

5. Let $\mathbf{F} = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$, the vector field from the take home problem.

   (a) (3 points) Use the Divergence Theorem to explain why $\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$ if S is the sphere of radius 1 centered at $(2, 2, 2)$.

   Solution: Let E be the region enclosed by S. Note that it doesn't contain the origin, so $\operatorname{div}\mathbf{F} = 0$ at every point of E. So $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\, dV = 0$.

   (b) (5 points) Find a function f defined everywhere except at the origin so that $\nabla f = \mathbf{F}$, or explain why no such function exists.

   Solution: $f(x, y, z) = -\dfrac{1}{(x^2+y^2+z^2)^{1/2}}$

   (c) (3 points) Explain why there is no vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}\mathbf{G}$.

   Solution: If there were such a $\mathbf{G}$, then for any surface S with boundary C, we know by Stokes' Theorem that

   $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{G}\cdot d\mathbf{S} = \int_C \mathbf{G}\cdot d\mathbf{r}$$

   But if S is a sphere centered at the origin, then since it has no boundary, this integral would be 0. But we know it is not 0, because it is 4π.

6. (15 points) [Take home problem] Let $\mathbf{F} = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$.

(a) Calculate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the sphere of radius a centered at the origin.

Solution: Parametrize the sphere by $\mathbf{r}(\phi,\theta) = \langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi\rangle$. Then

$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= a^2\langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \cos\phi\sin\phi\rangle\\ \mathbf{F} &= \frac{a\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle}{a^3}\end{aligned}$$

When we dot these, we get $\sin\phi$. Finally, we have $\displaystyle\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi$.

Another way (from Jebessa Dara's solution) is to use $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$. The unit normal vector is $\langle x, y, z\rangle/a$, which dots with $\mathbf{F}$ to give $(x^2+y^2+z^2)/a^4 = 1/a^2$. So the integral is

$$\int \frac{1}{a^2}\, dS = \frac{1}{a^2}(\text{surface area of } S) = 4\pi.$$

(b) Find $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$ at every point except the origin.

Solution: $\operatorname{curl}\mathbf{F} = \mathbf{0}$ and $\operatorname{div}\mathbf{F} = 0$.

(c) Explain why your answer to Part (a) doesn't depend on a.

Solution: If you have a sphere S_1 of radius a_1 and a larger sphere S_2 of radius a_2, then

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\, dV,$$

where E is the region between the two spheres. Since E does not contain the origin, $\operatorname{div}\mathbf{F} = 0$ on all of E, so the triple integral is 0. This explains why the two surface integrals are equal.

(d) Use Stokes' Theorem to explain why line integrals of $\mathbf{F}$ are independent of path.

Solution: Line integrals of $\mathbf{F}$ being path independent is the same as integrals of $\mathbf{F}$ along closed loops being 0. If you have an integral along a closed loop, make a surface S that has the loop as its boundary and make sure that the surface doesn't go through the origin. Then by Stokes' Theorem,

$$\int \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0.$$