Differential equation with discontinuities

$$y'' + y = \begin{cases} t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } 2\pi \le t \end{cases} \qquad \begin{aligned} y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

1. Convert piecewise to Heaviside
2. Laplace transform (shift before)
3. Simplify
4. Partial fractions
5. Inverse Laplace (shift after)
6. Convert Heaviside to piecewise

1. Convert piecewise to Heaviside

$$g(t) = \begin{cases} t & \text{if } 0 \leq t < 2\pi \\ 0 & \text{if } 2\pi \leq t \end{cases}$$

$$\begin{aligned} g(t) &= t[u_0(t) - u_{2\pi}(t)] \\ &= t[1 - u_{2\pi}(t)] \\ &= t - tu_{2\pi}(t) \end{aligned}$$

2. Laplace transform (shift before)

$$y'' + y = t - tu_{2\pi}(t), \qquad y(0) = 1, \qquad y'(0) = 1$$

$$\mathcal{L}\{tu_{2\pi}(t)\} = e^{-2\pi s}\mathcal{L}\{t + 2\pi\}$$

$$s^2Y - s - 1 + Y = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{2\pi}{s}\right)e^{-2\pi s}$$

3. Simplify

$$s^2 Y - s - 1 + Y = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{2\pi}{s}\right) e^{-2\pi s}$$

$$(s^2 + 1)Y = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{2\pi}{s}\right) e^{-2\pi s} + s + 1$$

$$(s^2 + 1)Y = \frac{s^3 + s^2 + 1}{s^2} - \frac{2\pi s + 1}{s^2} e^{-2\pi s}$$

$$Y = \frac{s^3 + s^2 + 1}{s^2(s^2 + 1)} - \frac{2\pi s + 1}{s^2(s^2 + 1)} e^{-2\pi s}$$

4. Partial fractions

$$\frac{s^3+s^2+1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1}$$

$$A(s^2+1) + Bs(s^2+1) + (Cs+D)s^2 = s^3+s^2+1$$

$$\begin{aligned} s = 0 &: A = 1 \\ s = i &: -(Ci+D) = -i \\ &\Longrightarrow\ C = 1,\ D = 0 \\ s = 1 &: 2A + 2B + C + D = 3 \\ &\Longrightarrow\ B = 0 \end{aligned}$$

4. Partial fractions again

$$\frac{2\pi s+1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1}$$

$$A(s^2+1) + Bs(s^2+1) + (Cs+D)s^2 = 2\pi s + 1$$

$$\begin{aligned}
s = 0 &: A = 1 \\
s = i &: -(Ci + D) = 2\pi i + 1 \\
&\Longrightarrow\ C = -2\pi i,\ D = -1 \\
s = 1 &: 2A + 2B + C + D = 2\pi + 1 \\
&\Longrightarrow\ B = 2\pi
\end{aligned}$$

5. Inverse Laplace transform (shift after)

$$Y = \frac{1}{s^2} + \frac{s}{s^2+1} - \left(\frac{1}{s^2} + \frac{2\pi}{s} - \frac{2\pi s + 1}{s^2+1}\right) e^{-2\pi s}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{2\pi}{s} - \frac{2\pi s + 1}{s^2+1}\right\} = t + 2\pi - 2\pi\cos(t) - \sin(t)$$

$$\begin{aligned} y = {} & t + \cos(t) \\ & - \left[t - 2\pi\cos(t - 2\pi) - \sin(t - 2\pi)\right] u_{2\pi}(t) \end{aligned}$$

6. Convert Heaviside to piecewise

$$y = t + \cos t - (t - 2\pi \cos t - \sin t)\, u_{2\pi}(t)$$

$$y = \begin{cases} t + \cos t & \text{if } 0 \le t < 2\pi \\ (2\pi + 1)\cos t + \sin t & \text{if } 2\pi \le t \end{cases}$$